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Discrete Mathematics 176 (1997) 223–231

DISCRETE MATHEMATICS

Permutations of the positive integers with specified differences

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Received 13 October 1993; revised 29 February 1996

Abstract

In this paper, we show that given any finite set, $D = \{D_1, D_2, \ldots, D_n\}$, of positive integers, with gcd $(D_1, D_2, \ldots, D_n) = 1$, there is a permutation of the positive integers such that the absolute value of the difference between any two consecutive values is in D. Further, it is possible to choose the permutation so that each element of D occurs infinitely often as a difference. This answers in the affirmative a conjecture of Slater and Velez (1977, 1979).

1. Introduction

Call a sequence of positive integers $\{a_i\}_{i=1}^{\infty}$ such that every positive integer occurs exactly once in the sequence, a permutation of the positive integers. For any such sequence, we get a sequence of differences $d_i = |a_{i+1} - a_i|$. Such sequences have been studied by Slater and Velez [1, 2]. In particular, they posed the following question [3]. Suppose $D = \{D_1, \ldots, D_n\}$ is a finite set of positive integers with $\gcd(D_1, \ldots, D_n) = 1$. Is there a permutation of the positive integers $\{a_i\}_{i=1}^{\infty}$ such that $|a_{i+1} - a_i| \in D$ for all $i \geqslant 1$? (Clearly, the condition $\gcd(D_1, \ldots, D_n) = 1$ is necessary.) In this paper, we will show that such a permutation always exists and furthermore that it is possible to choose this permutation so that each D_j occurs infinitely often.

Our method will be very similar to that employed in [2]. The first step is to build long paths involving only two of the elements. In [2], this was done only for pairs of relatively prime elements. We shall allow the two elements to have a nontrivial greatest common divisor. As a result, we will be forced to require more conditions on our paths. Also in [2] the basic building block for paths was a short standard path. Our basic building block will be a (very similar) short cycle instead. For greater flexibility in constructions, we will often regard sequences and cycles of positive integers as graph theoretic objects, specifically paths and cycles in $K(\mathbb{Z})$, the complete graph on the set of integers $\mathbb{Z}$. In graph theoretic terms, a permutation of the positive integers is just a hamiltonian path in $K(\mathbb{Z}^+)$. The statement that all differences of the

PII S0012-365X(96)00154-9

permutation are in a fixed set is equivalent to saying that all the lengths of edges in the hamiltonian path are in that set. If m and n are integers, we let $[m, n]$ denote the edge joining them. If C is a cycle containing the edge $e = [m, n]$, then we let $C \backslash e$ denote the (undirected) path joining m and n that one obtains by removing the edge e from C. If X is any subset of $K(\mathbb{Z})$, a is any positive integer, and b is any integer, then we let $aX + b$ be the result of applying the affine transformation $f(x) = ax + b$ pointwise to X.

Section 2 will be devoted to constructing, for any pair of relatively prime positive integers $b > a \geqslant 1$, a collection of paths with all the edge lengths equal to a or b. The constructions are slightly technical. Section 3 is devoted to applying these constructions to prove Slater and Velez's conjecture. Roughly we choose two of our allowed differences D_1 and D_2. We then apply the constructions to $a = D_1/g$ and $b = D_2/g$, where $g = \gcd(D_1, D_2)$. Our permutation of the positive integers is then built out of pieces $gX + r$, where X is one of the paths constructed in Section 2.

2. Constructions with two differences

This section is devoted to the proof of the following proposition.

Proposition 1. *Let a and b be relatively prime integers with $b > a \geqslant 1$ and suppose $N = q(a + b) + r$, where $q \geqslant 0$ and $0 \leqslant r < a + b$. If either $r < b$ or $r \geqslant 2a$, then there is a path $x_0, x_1, \dots, x_M$ such that*

(1) $x_0 = N$,

(2) $|x_{i+1} - x_i| = a$ *or* b,

(3) $\{x_0, x_1, \dots, x_M\} = \{0, 1, \dots, M\}$,

(4) *If $r < b$ and we write the terminal endpoint as $x_M = s(a + b) + t$, where $0 \leqslant t < a + b$, then t may be arranged to be any element of $\{a, \dots, a + b - 1\}$ congruent to $r + a$ mod $b - a$, and*

(5) *If $r \geqslant 2a$ and we write the terminal endpoint as $x_M = s(a + b) + t$, where $0 \leqslant t < a + b$, then t may be arranged to be any element of $\{a, \dots, a + b - 1\}$ congruent to $r - a$ mod $b - a$.*

Furthermore, in either case, we may assume s and hence x_M is arbitrarily large.

Proof. To build these examples we will construct successively more complicated objects. The first of these is the basic cycle which is fundamental to all our constructions.

Step 1: Define a cycle $C_1(a, b) = \{x_0, x_1, \dots, x_{a+b-1}, x_{a+b} = x_0\}$ by letting x_i be the unique element of $\{0, 1, \dots, a + b - 1\}$ which is congruent to ai mod $a + b$. (Alternately we may define x_i inductively by $x_0 = 0$ and x_{i+1} is whichever of $x_i + a$ or $x_i - b$ is still in $\{0, 1, \dots, a + b - 1\}$.) See Fig. 1. Note that $C_1(a, b)$ has the following properties.

(1) $|x_{i+1} - x_i| = a$ or b,

(2) $\{x_0, x_1, \dots, x_{a+b-1}\} = \{0, 1, \dots, a + b - 1\}$, and

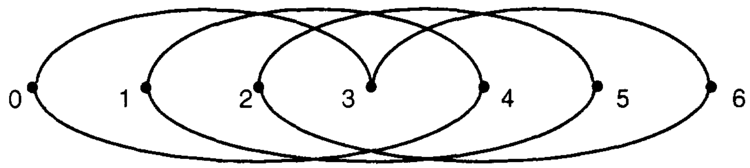


Fig. 1. The basic cycle $C_1(3, 4)$.

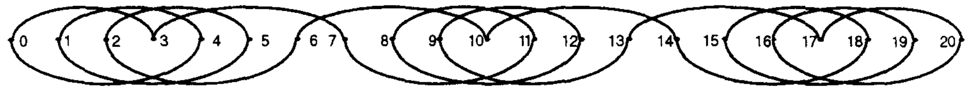


Fig. 2. The cycle $C_3(3, 4)$.

(3) x and y in $\{0, 1, \dots, a+b-1\}$ are adjacent in $C_1(a, b)$ if and only if $x - y \equiv \pm a \pmod{a+b}$.

Step 2: For any $r > 1$ we can build a cycle r times as long by tying together, r translates of $C_1(a, b)$. Explicitly start with the adjacent translates $C_1(a, b)$, $C_1(a, b) + (a+b), \dots, C_1(a, b) + (r-1)(a+b)$. Remove the edges

$$[a, 2a], [a, 2a] + (a+b), \dots, [a, 2a] + (r-2)(a+b).$$

(one from each of the first $r-1$ translates) and the edges

$$[0, a] + (a+b), [0, a] + 2(a+b), \dots, [0, a] + (r-1)(a+b).$$

(one from each of the last $r-1$ translates.) Then add the edges

$$[a, a+b], \dots, [a, a+b] + (r-2)(a+b)$$

and the edges

$$[2a, 2a+b], \dots, [2a, 2a+b] + (r-2)(a+b).$$

See Fig. 2. Denote the result by $C_r(a, b)$. To see that $C_r(a, b)$ is a cycle, note that in going from $C_{r-1}(a, b)$ to $C_r(a, b)$ we remove one edge, $[a, 2a] + (r-2)(a+b)$, from $C_{r-1}(a, b)$ and one edge, $[0, a] + (r-1)(a+b)$, from the translate $C_1(a, b) + (r-1)(a+b)$. This leaves us with two paths. We then glue in the two edges $[0, a] + (r-1)(a+b)$ and $[a, a+b] + (r-2)(a+b)$ joining up the four endpoints of the two paths. Thus, the result is again a cycle.

Write $C_r(a, b) = \{x_0, x_1, \dots, x_{r(a+b)-1}, x_{r(a+b)} = x_0\}$. Note that $C_r(a, b)$ has the following properties.

(1) $|x_{i+1} - x_i| = a$ or b,

(2) $\{x_0, x_1, \dots, x_{r(a+b)-1}\} = \{0, 1, \dots, r(a+b)-1\}$, and

(3) $C_r(a, b)$ contains every edge of $C_1(a, b) + (r-1)(a+b)$ except $[0, a] + (r-1)(a+b)$.

Step 3: At least part of the proposition is now immediate since deleting any edge from the cycle $C_r(a, b)$ will give us a path. Recall that $N = q(a + b) + r$, where $0 \leqslant r < a + b$, and suppose first that $0 < r < b$. Then by construction $C_{q+1}(a, b)\backslash[N, N + a]$ is a path of length $(q + 1)(a + b)$ from $x_0 = N$ to $x_M = q(a + b) + r + a$ which passes through every vertex in $\{0, 1, \dots, (q + 1)(a + b) - 1\}$. Note that the remainder $\bmod\, a + b$ of the terminal endpoint x_M has the correct residue $\bmod\, b - a$. Also note one additional fact, $x_M + b \geqslant (q + 1)(a + b)$, i.e., stepping right a distance b from the end of the path takes us past the construction done so far. If $r = 0$, then the edge $[N, N + a]$ is not in $C_{q+1}(a, b)$. One can instead take the path $C_{q+1}(a, b)\backslash[N, N + b]$. The terminal endpoint $x_M = q(a + b) + b = q(a + b) + a + (b - a)$ still has an acceptable remainder and we still have $x_M + b \geqslant (q + 1)(a + b)$.

Now suppose that $r \geqslant 2a$. Then consider the path $C_{q+1}(a, b)\backslash[N - a, N]$. Again this is a path of length $(q + 1)(a + b)$ from $x_0 = N = q\ (a + b) + r$ to, in this case, $x_M = q(a + b) + r - a$ which passes through every vertex in $\{0, 1, \dots, (q + 1)(a + b) - 1\}$. The remainder mod $a + b$ of the terminal endpoint x_M is congruent to $r - a$ mod $b - a$ as claimed and again $x_M + b = q(a + b) + r + b - a \geqslant (q + 1)(a + b)$.

These are some of the paths required. It remains to be shown that the endpoint x_M can be assumed to be large and that the remainder of x_M mod $a + b$ can be varied as claimed. Both these are relatively easy and do not require any of the details of the constructions above except the properties of $C_1(a, b)$. For the first extension the following lemma clearly suffices.

Lemma 1. *Suppose $P = \{x_0, x_1, \dots, x_{s(a+b)-1}\}$ is a path going through the vertices $\{0, 1, \dots, s(a + b) - 1\}$ and suppose $x_{s(a+b)-1} + b \geqslant s(a + b)$. Then there is a path $P' = \{x_0, x_1, \dots, x_{(s+1)(a+b)-1}\}$ with the following properties*

(1) $|x_i - x_{i-1}| = a$ or b, for $i = s\,(a + b), \dots, (s + 1)(a + b) - 1$,

(2) $\{x_0, x_1, \dots, x_{(s+1)(a+b)-1}\} = \{0, 1, \dots, (s + 1)(a + b) - 1\}$, *and*

(3) $x_{(s+1)(a+b)-1} = x_{s(a+b)-1} + a + b$.

Proof. Set

$$P' = P \cup [x_{s(a+b)-1}, x_{s(a+b)-1} + b] \cup ((C_1(a, b) + s(a + b))$$

$$\backslash[x_{s(a+b)-1} + b, x_{s(a+b)-1} + a + b]).$$

See Fig. 3. □

Applying this lemma repeatedly we may clearly arrange that x_M is large while keeping all differences in $\{a, b\}$ and not changing $x_M \bmod a + b$. To complete the proof of Proposition 1 we only need to show that we can change the remainder of x_M mod $a + b$ by adding or subtracting $b - a$. These are very similar to Lemma 1.

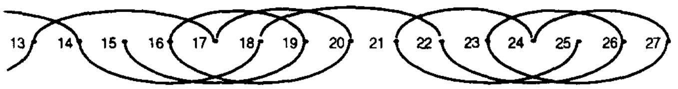
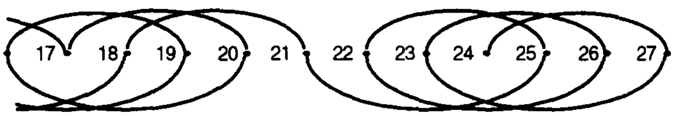


Fig. 3. The path $C_3(3,4)\backslash[15,18]$ extended once as described in Lemma 1. (The portion not shown is identical to the same portion of Fig. 2.)



(a)

Fig. 4. The path $C_3(3,4)\backslash[15,18]$ extended once as described in Lemma 2. (The portion not shown is as in Fig. 3.)

Lemma 2. *Suppose* $P = \{x_0, x_1, \dots, x_{s(a+b)-1}\}$ *is a path going through the vertices* $\{0, 1, \dots, s(a+b)-1\}$ *and suppose* $x_{s(a+b)-1} = (s-1)(a+b)+r$, *where* $r \geqslant b$. *Then there is a path* $P' = \{x_0, x_1, \dots, x_{(s+1)(a+b)-1}\}$ *with the following properties*

(1) $|x_i - x_{i-1}| = a$ *or* b, *for* $i = s(a+b), \dots, (s+1)(a+b)-1$,

(2) $\{x_0, x_1, \dots, x_{(s+1)(a+b)-1}\} = \{0, 1, \dots, (s+1)(a+b)-1\}$, *and*

(3) $x_{(s+1)(a+b)-1} = x_{s(a+b)-1} + 2a = x_{s(a+b)-1} + (a+b) - (b-a)$.

Proof. Set

$$P' = P \cup [x_{s(a+b)-1}, x_{s(a+b)-1} + a] \cup ((C_1(a,b) + s(a+b)) \backslash [x_{s(a+b)-1} + a, x_{s(a+b)-1} + 2a]).$$

See Fig. 4. □

Lemma 3. *Suppose* $P = \{x_0, x_1, \dots, x_{s(a+b)-1}\}$ *is a path going through the vertices* $\{0, 1, \dots, s(a+b)-1\}$ *and suppose* $x_{s(a+b)-1} = (s-1)(a+b)+r$, *where* $r < 2a$. *Then there is a path* $P' = \{x_0, x_1, \dots, x_{(s+1)(a+b)-1}\}$ *with the following properties*

(1) $|x_i - x_{i-1}| = a$ *or* b, *for* $i = s(a+b), \dots, (s+1)(a+b)-1$,

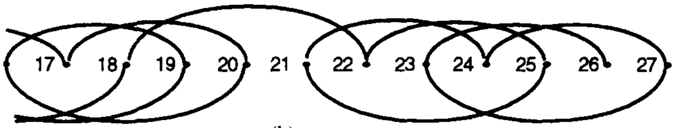
(2) $\{x_0, x_1, \dots, x_{(s+1)(a+b)-1}\} = \{0, 1, \dots, (s+1)(a+b)-1\}$, *and*

(3) $x_{(s+1)(a+b)-1} = x_{s(a+b)-1} + 2b = x_{s(a+b)-1} + (a+b) + (b-a)$.

Proof. Set

$$P' = P \cup [x_{s(a+b)-1}, x_{s(a+b)-1} + b] \cup ((C_1(a,b) + s(a+b)) \backslash [x_{s(a+b)-1} + b, x_{s(a+b)-1} + 2b]).$$

See Fig. 5. □



(b)

Fig. 5. The path $C_3(3, 4)\backslash[15, 18]$ extended once as described in Lemma 3. (The portion not shown is as in Fig. 3.)

Applying Lemmas 2 and 3 repeatedly, we may clearly change the remainder $\mathrm{mod}(a + b)$ of the endpoint x_M by adding or subtracting $b - a$. This allows us to produce any remainder in the interval $\{a, a + 1, \ldots, a + b - 1\}$ in the required congruence class. Therefore, the proof of Proposition 1 is complete. □

Denote the path, guaranteed by Proposition 1, which begins at N, ends at x_M and has all steps of length a or b by $P(a, b; N, x_M)$.

3. Proof of the conjecture

The aim of this section is to prove the following affirmative answer to a conjecture of Slater and Velez [1, 2].

Theorem 1. *Let $D = \{D_1, \ldots, D_n\}$ be a finite set of positive integers with $\gcd(D_1, \ldots, D_n) = 1$. Then there is a permutation of the positive integers $\{a_i\}_{i=1}^{\infty}$ such that $|a_{i+1} - a_i| \in D$ for all $i \geqslant 1$.*

Proof. Choose two of the differences, without loss of generality, take them to be D_1 and D_2 with $D_2 > D_1$. Let $g = \gcd(D_1, D_2)$. Let $a = D_1/g$ and $b = D_2/g$. Choose a sequence $\{p_i\}_{i=1}^{\infty}$ with the following properties. For all i, $|p_i| \in D$ and p_i is not divisible by g, and every congruence class mod g occurs infinitely often in the sequence $\{s_i = p_1 + \cdots + p_i\}$ of partial sums of $\{p_i\}$.

The outline of the construction is as follows. We think of the positive integers as being divided up into congruence classes mod g. Imagine each congruence class as a column of elements $r, r + g, r + 2g, \ldots$. We wish to build our permutation by first filling one column up to a certain point, making sure not to leave any gaps. We then use p_1 to move to another column. We then fill that column up to a much higher point, again leaving no gaps. We then use p_2 to move to a third column, fill that congruence class to a much higher point, and move out using p_3, etc. The final result will be the desired permutation of the positive integers.

Filling the first column is easy. For definitiveness, we will begin in the congruence class of 0 mod g. For any edge e of $C_r(a, b)$, $P' = g[C_r(a, b)\backslash e] + g$ is a long path with the following properties. All the differences of P' are either D_1 or D_2. Also P' lies entirely in the congruence class of 0 mod g and P' goes through an initial block of consecutive elements of that congruence class. Note that, by choosing r large, we make P' use as many elements as we like. Also, by choosing e correctly and orienting the path correctly, we may choose any number divisible by g to be the terminal endpoint x' of P'. At the next step in the construction, we will step out of the congruence class with a move of p_1. This will place us at the point $x' + p_1$. Write this point as $x' + p_1 = gq + r$ for some $1 \leqslant r < g$. Since x' was arbitrary, we may arrange that $x' + p_1$ is positive and that q is a multiple of $a + b$. Fix one such set of choices and call the resulting path Q_0 and its terminal endpoint y_0.

We now wish to fill up the congruence class of r mod g up to some large point starting at $gq + r$. The paths given by Proposition 1 are almost ideal for filling up the columns. Suppose $P = P(a, b; N, x_M)$ is one of the paths given by Proposition 1. Then $gP + k$ (for k any positive integer) is a path from $gN + k$ to $gx_M + k$ all of whose differences are either D_1 or D_2. Also this path now goes through a consecutive stretch of integers, beginning with k, within a single congruence class mod g. To fill up the second congruence class we can use the path $g\,P(a, b; q, x_M) + r$ for any x_M. This path exists by Proposition 1 since we arranged q to be a multiple of $a + b$. For any choice of x_M, the union $P' = Q_0 \cup [y_0, y_0 + p_1] \cup (gP(a, b; q, x_M) + r)$ is a path with terminal endpoint $x' = gx_M + r$ which has the following properties. Every difference of P' is in D and

(P1) For each j, $1 \leqslant j \leqslant g$, there is a nonnegative integer k_j such that

$$P' \cap \{j, g + j, 2g + j, \dots\} = \{j, g + j, \dots, g\,(k_j - 1) + j\},$$

that is, P' uses up a contiguous (possibly empty) initial block in each congruence class mod g. By Proposition 1, we have a great deal of freedom in our choice of the terminal endpoint x'. However, there are two conditions which we will need to impose on x' in order to be able to carry out the next stage in the construction.

For the next stage in the construction, we want to step by the edge $[x', x' + p_2]$. Suppose this places us in the congruence class of j mod g for $1 \leqslant j \leqslant g$. Then the smallest element of this congruence class not visited by P' is $gk_j + j$. (Note that k_j is independent of the choice of the terminal endpoint for P' since g does not divide p_2.) To ensure that we still have a path, we need to have x' large enough that the following condition holds.

(P2) $x' + p_2 \geqslant gk_j + j$.

By the remark at the end of Proposition 1, this can easily be arranged. Write $x' + p_2 = gq' + j$. Then we would like to extend P' by forming the union $P' \cup [x', x' + p_2] \cup (gP(a, b;\ q' - k_j, x_M) + gk_j + j)$. The path $gP(a, b;\ q' - k_j, x_M) + gk_j + j$ will go through a contiguous block in the congruence class of j mod g beginning with $gk_j + j$. Therefore, this union will have property P1 defined above. Unfortunately, Proposition 1 need not guarantee the existence of the path $P(a, b; q' - k_j, x_M)$. Write $q' - k_j$ as $q' - k_j = \theta(a + b) + \rho$ for $0 \leqslant \rho < a + b$. Then the path $P(a, b;\ q' - k_j, x_M)$ exists if either $\rho < b$ or $\rho \geqslant 2a$. Therefore, we also need x' to satisfy the following additional condition.

(P3) Write $(x' + p_2 - j)/g - k_j = \theta(a+b) + \rho$ with $0 \leqslant \rho < a+b$,

then either $\rho < b$ or $\rho \geqslant 2a$.

If $b \geqslant 2a$, then condition P3 holds trivially. Suppose that $b < 2a$. We will show that we can use the flexibility, given to us by Claims (4) and (5) of Proposition 1, to choose an x' for which P3 holds. First, recall the freedom given to us by these two claims. Suppose we write the terminal endpoint x_M of the path $P(a, b;\ q, x_M)$ as $x_M = s(a+b) + t$. By Proposition 1, t may be arranged to be any element of $\{a, \dots, a+b-1\}$ in a fixed congruence class mod $b-a$. Denote these values by t_0, $(b-a) + t_0, \dots, m(b-a) + t_0$ and note that $m + 1 \geqslant \lfloor b/(b-a) \rfloor$, where $\lfloor x \rfloor$ denotes the greatest integer less than or equal to x. If $x_M = s(a+b) + i(b-a) + t_0$, then $x' = gs(a+b) + ig(b-a) + gt_0 + r$ and $q' - k_j = s(a+b) + i(b-a) + [t_0 + (r + p_2 - j)/g - k_j]$. From this formula, we see that the possible values for $(x' + p_2 - j)/g - k_j = q' - k_j$ form an arithmetic progression of length at least $\lfloor b/(b-a) \rfloor$ with steps of size $b-a$. Write each of these possible values of $q' - k_j$ as $s(a+b) + i(b-a) + [t_0 + (r + p_2 - j)/g - k_j] = \theta_i(a+b) + \rho_i$. Condition P3 holds for some choice of x' if one of these remainders ρ_i is not in the excluded interval $b \leqslant \rho < 2a$. Since $\rho_i \equiv \rho_{i-1} + (b-a) \pmod{a+b}$, we see that if $\rho_{i-1} < 2a$ then $\rho_i = \rho_{i-1} + (b-a)$, and if $\rho_{i-1} \geqslant 2a$ then $\rho_i = \rho_{i-1} - 2a$. We can paraphrase this remark as follows. As we increase i, the remainders ρ_i also form an arithmetic progression with steps of size $b-a$ until we reach a value of ρ_i satisfying $\rho_i \geqslant 2a$. Thus, the only way that all of the ρ_i can lie in the excluded interval is if they all form a single arithmetic progression with steps of size $b-a$ inside the excluded interval $b \leqslant \rho < 2a$. However, the excluded interval can contain an arithmetic progression of length at most $\lceil (2a-b)/(b-a) \rceil$ with steps of size $b-a$, where $\lceil x \rceil$ denotes the least integer greater than or equal to x. Further, one easily sees that $\lfloor b/(b-a) \rfloor = \lfloor (2a-b)/(b-a) \rfloor + 2 > \lceil (2a-b)/(b-a) \rceil$. Hence, some ρ_i is not excluded and we can always choose P' and x' so that condition P3 holds. Fix one such choice and call this path Q_1 and its terminal endpoint y_1.

By essentially the same argument, it follows that we can inductively build a nested sequence of paths Q_i with terminal endpoint y_i with the following properties. Every difference in Q_i is in D, and

(P1) For each j, $1 \leqslant j \leqslant g$, there is a nonnegative integer k_j such that

$$Q_i \cap \{j, g+j, 2g+j, \dots\} = \{j, g+j, \dots, g(k_j - 1) + j\}.$$

If $y_i + p_{i+1} \equiv l \pmod g$ for $1 \leqslant l \leqslant g$, then

(P2) $y_i + p_{i+1} \geqslant gk_l + l$, and

(P3) Write $(y_i + p_{i+1} - l)/g - k_l = \theta(a+b) + \rho$ with $0 \leqslant \rho < a+b$,

then either $\rho < b$ or $\rho \geqslant 2a$.

Given Q_i, we write $y_i + p_{i+1} = gq + j$ and take

$$P' = Q_i \cup [y_i, y_i + p_{i+1}] \cup (gP(a, b; q - k_j, x_M) + gk_j + j).$$

The path $P(a, b; q - k_j, x_M)$ exists by Proposition 1 since Q_i satisfies P3 and the union is a path since Q_i satisfies P2. As we argued above, P' is again a path with all differences in D and P' has property P1 for any choice of x_M. The terminal endpoint of P' is $x' = gx_M + gk_j + j$. By choosing x_M large, we can arrange that P' satisfies property P2. Also, as argued above, we may choose x_M to be any value in an arithmetic progression of length $\lfloor b/(b - a) \rfloor$ with steps of length $b - a$. Further, as above, at least one of these choices must give a value of x' which satisfies P3. Fix one such choice of x_M, call the resulting path Q_{i+1}, and call its terminal endpoint y_{i+1}.

The union of the Q_i is then our desired permutation of the positive integers. $\square$

In their paper [1], Slater and Velez also ask whether one can choose the permutation so that every element of D occurs infinitely often as a difference. The construction above can be slightly modified to get this additional property. Choose $g = \max_{i,j} \gcd(D_i, D_j)$. Note that if g divides D_i and D_j, then by maximality of g we must have $\gcd(D_i, D_j) = g$. Reorder D so that D_1 is the largest element of D divisible by g and $\{D_2, \ldots, D_r\}$ are the other elements of D divisible by g. The constructions above can be done with any pair $\{D_i, D_1\}$, $2 \leqslant i \leqslant r$. In particular, the extension procedure used in Lemma 1 can be done with any such pair, and that pair can even be changed within a single path. Therefore, we merely add the following conditions to our constructions above.

(1) Build our paths with Lemma 1 using predominantly $\{D_1, D_2\}$, but arrange that each pair $\{D_i, D_1\}$, $3 \leqslant i \leqslant r$, is used infinitely often in the extensions of paths.

(2) Arrange that every element of D which is not divisible by g is used infinitely often in $\{p_i\}$. Thus, we have the following addendum to the previous theorem.

Addendum. It is further possible to choose $\{a_i\}_{i=1}^{\infty}$ so that every element of D occurs infinitely often in the sequence $\{d_i = |a_{i+1} - a_i|_{i=1}^{\infty}$.

Acknowledgements

The author was partially supported by an Alfred P. Sloan Fellowship and an NSF Postdoctoral Fellowship.

References

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